Flow of a Rarefied Gas between Parallel and Almost Parallel Plates

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Abstract. Rarefied gas flows in ultra-thin film slider bearings are studied in a wide range of Knudsen numbers. The generalized Reynolds equation first derived by Fukui and Kaneko [1], [2] has been extended by allowing for bounding surfaces with different physical structures, as an issue of relevance for applications. Since the solution of this equation requires that the Poiseuille and Couette flow rates between two parallel plates have to be accurately calculated in advance, we have used our recent results on Poiseuille flow [3], [4] and new results on Couette flow to evaluate the lubrication characteristics.

FLOW OF A RAREFIED GAS BETWEEN TWO PARALLEL PLATES: THE POISEUILLE-COUETTE PROBLEM

Let us consider two plates separated by a distance h and a gas flowing parallel to them, in the x direction, due to a pressure gradient. The lower boundary (placed at $z = -h/2$) moves to the right with velocity U, while the upper boundary (placed at $z = h/2$) is fixed. Both boundaries are held at a constant temperature T_o.

If the pressure gradient is taken to be small as well as the velocity U, it can be assumed that the velocity distribution of the flow is nearly the same as that occurring in an equilibrium state. This means that the Boltzmann equation can be linearized about a Maxwellian f_0 by putting [5]:

$$f = f_0(1+h) \tag{1}$$

where $f(x,z,\mathbf{c})$ is the distribution function for the molecular velocity $\mathbf{c}$ expressed in units of $(2RT_o)^{1/2}$ (R being the gas constant), z is the coordinate normal to the plates and $h(z,\mathbf{c})$ is the small perturbation upon the basic equilibrium state. If one assumes the linearized BGK model for the collision operator [6], the Boltzmann equation reads [7]:

$$\frac{1}{2}k + c_z\frac{\partial Z}{\partial z} = \frac{1}{\theta}\left[\pi^{-\frac{1}{2}}\int_{-\infty}^{+\infty}e^{-c_{z_1}^2}Z(z,c_{z_1})\,dc_{z_1} - Z(z,c_z)\right] \tag{2}$$

where by definition

$$Z(z,c_z) = \pi^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{-c_x^2-c_y^2}c_x h(z,\mathbf{c})\,dc_x dc_y$$

$$k = \frac{1}{p}\frac{\partial p}{\partial x} = \frac{1}{\rho}\frac{\partial \rho}{\partial x}$$

with p and ρ being the gas pressure and density, respectively, and θ is the collision time. Consequently, the integral equation for the bulk velocity of the gas can be written as follows:

$$q(z) = \pi^{-\frac{1}{2}}\int_{-\infty}^{+\infty}e^{-c_{z_1}^2}Z(z,c_{z_1})\,dc_{z_1} \tag{3}$$

From Eq. (2) we obtain in integral form [3], [8]:

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$$Z(z,c_z) = \quad \exp(-(z+\tfrac{h}{2}\mathrm{sgn}c_z)/(c_z\theta))Z(-\tfrac{h}{2}\mathrm{sgn}c_z,c_z) +$$

$$\int_{-\frac{h}{2}sgnc_z}^{z} \exp\left(\frac{-|z-t|}{|c_z|\theta}\right)[q(t)-k\theta/2]/(c_z\theta)\,dt \tag{4}$$

with the values at the boundary, $Z(-\tfrac{h}{2}\mathrm{sgn}c_z,c_z)$, depending on the model of boundary conditions chosen. In the following, we will consider the Maxwell boundary conditions and specialize the analysis to walls having different physical properties so that two accommodation coefficients (α_1,α_2) can be defined. In this case, the boundary conditions can be written as [3], [8]:

$$Z^+(h/2,c_z) = (1-\alpha_1)Z^-(h/2,-c_z) \tag{5}$$

$$Z^+(-h/2,c_z) = \alpha_2 U + (1-\alpha_2)Z^-(-h/2,-c_z) \tag{6}$$

where U is expressed in units of $(2RT_o)^{1/2}$; $Z^-(-h/2,c_z)$, $Z^-(h/2,c_z)$ are the distribution functions of the molecules impinging upon the walls and $Z^+(-h/2,c_z)$, $Z^+(h/2,c_z)$ the distribution functions of the molecules reemerging from them.

Once the function at the boundary, $Z(-\tfrac{h}{2}\mathrm{sgn}c_z,c_z)$, has been evaluated following the analytical procedure reported in [3], [4], the substitution of the integral formula (4) in the definition (3) of $q(z)$ gives the integral equation for the bulk velocity of the gas:

$$q(z) = \frac{1}{2}k\theta[1-\psi_p(u)] + U\psi_c(u) \tag{7}$$

Eq. (7) shows that the gas velocity is induced by the superposition of two distinct effects. The gas moves by an imposed pressure gradient (Poiseuille flow) and by the shear driven flow due to the motion of the bottom surface (Couette flow). The non-dimensional functions $\psi_p(u)$ and $\psi_c(u)$, giving the Poiseuille and Couette contributions, respectively, can be written as:

$$\psi_p(u) = \quad 1 + \frac{1}{\sqrt{\pi}}\int_{-\delta/2}^{\delta/2} dw\,\psi_p(w)\Big\{(1-\alpha_1)S_{-1}(\delta-u-w) + (1-\alpha_2)S_{-1}(\delta+u+w) +$$

$$(1-\alpha_1)(1-\alpha_2)[S_{-1}(2\delta-u+w)+S_{-1}(2\delta+u-w)] + T_{-1}(|u-w|)\Big\} \tag{8}$$

$$\psi_c(u) = \quad \frac{\alpha_2}{\sqrt{\pi}}\Big[T_o(\delta/2+u) + (1-\alpha_1)S_o(3/2\delta-u) + (1-\alpha_1)(1-\alpha_2)S_o(5/2\delta+u)\Big] +$$

$$\frac{1}{\sqrt{\pi}}\int_{-\delta/2}^{\delta/2} dw\,\psi_c(w)\Big\{(1-\alpha_1)S_{-1}(\delta-u-w) + (1-\alpha_2)S_{-1}(\delta+u+w) +$$

$$(1-\alpha_1)(1-\alpha_2)[S_{-1}(2\delta-u+w)+S_{-1}(2\delta+u-w)] + T_{-1}(|u-w|)\Big\} \tag{9}$$

where $T_n(x)$ is the Abramowitz function defined by

$$T_n(x) = \int_0^{+\infty} t^n \exp(-t^2 - x/t)\,dt$$

$S_n(x)$ is a generalized Abramowitz function defined by

$$S_n(x,\delta,\alpha_1,\alpha_2) = \int_0^{+\infty} \frac{t^n\exp(-t^2-x/t)}{1-(1-\alpha_1)(1-\alpha_2)\exp(-2\delta/t)}\,dt$$

and the following non-dimensional variables have been introduced:

$$\delta = h/\theta, \quad w = t/\theta, \quad u = z/\theta.$$
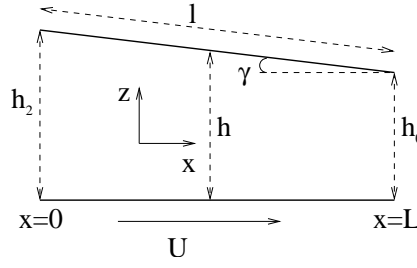
FIGURE 1. Geometry of a slider bearing.

Using Eq. (7), the flow rate (per unit time through unit thickness) defined by [7]:

$$F = \rho \int_{-h/2}^{h/2} q(z)dz \tag{10}$$

can be expressed as the sum of the Poiseuille flow (F_p) and the Couette flow (F_c) as follows:

$$F = F_p + F_c = -\frac{\partial p}{\partial x} h^2 Q_p(\delta, \alpha_1, \alpha_2) + \frac{\rho U h}{2} Q_c(\delta, \alpha_1, \alpha_2) \tag{11}$$

where

$$Q_p(\delta, \alpha_1, \alpha_2) = -\frac{1}{\delta} + \frac{1}{\delta^2} \int_{-\delta/2}^{\delta/2} \psi_p(u)du$$

$$Q_c(\delta, \alpha_1, \alpha_2) = \frac{2}{\delta} \int_{-\delta/2}^{\delta/2} \psi_c(u)du$$

are the non-dimensional volume flow rates.

THE GENERALIZED REYNOLDS EQUATION

The analysis developed in the previous section can be applied to the slider bearing problem in lubrication theory. The basic geometry of the two-dimensional gas film is outlined in Fig. 1. Unlike the configuration considered in the previous section, the upper plate is slightly inclined at a small angle γ. Since the pitch angle γ is typically less than 1°, the pressure p in the gas is taken to be constant in the vertical direction, while at the left ($x = 0$) and right ($x = L$) boundaries it is fixed at ambient pressure p_o.

In standard hydrodynamic lubrication theory, the pressure distribution in the gas film can be computed from the classical Reynolds equation (an approximation of the full Navier-Stokes equation):

$$\frac{d}{dX} \left(PH^3 \frac{dP}{dX} - \Lambda PH \right) = 0 \tag{12}$$

where the no-slip boundary conditions for the velocities at the bottom and top surfaces have been used. Equation (12) is written in terms of the following dimensionless quantities

$$X = \frac{x}{l}; \quad P = \frac{p}{p_o}; \quad H = \frac{h}{h_o} \tag{13}$$

and the bearing number is defined as

$$\Lambda = \frac{6\mu U l}{p_o h_o^2} \tag{14}$$

where μ is the dynamic viscosity of the gas.

To extend the generalized Reynolds equation, the mass flow conservation equation across the film thickness must be applied [1], [2], with the flow rate F given by Eq. (11):

$$\frac{d}{dx}\left(\frac{dp}{dx}h^2 Q_p(\delta,\alpha_1,\alpha_2) - \frac{\rho U h}{2}Q_c(\delta,\alpha_1,\alpha_2)\right) = 0 \tag{15}$$

For the purpose of a direct comparison with the classical Reynolds equation (12), let us introduce the Poiseuille relative flow rate [1], [2]:

$$\tilde{Q}_p(\delta,\alpha_1,\alpha_2) = \frac{Q_p(\delta,\alpha_1,\alpha_2)}{Q_{con}} \tag{16}$$

where $Q_{con} = \delta/6$. Furthermore, the rarefaction parameter δ can be expressed as: $\delta = \delta_o PH$, where δ_o is the characteristic inverse Knudsen number defined by the minimum film thickness, h_o, and the ambient pressure p_o as:

$$\delta_o = \frac{p_o h_o}{\mu\sqrt{2RT_o}}$$

Finally, assuming that the heat generation in the gas is very small, so that an isothermal process can be considered, the non-dimensional generalized Reynolds equation reads:

$$\frac{d}{dX}\left(\tilde{Q}_p(\delta_o PH,\alpha_1,\alpha_2)PH^3\frac{dP}{dX} - Q_c(\delta_o PH,\alpha_1,\alpha_2)\Lambda PH\right) = 0 \tag{17}$$

If the two walls are identical ($\alpha_1 = \alpha_2 = \alpha$), the Couette flow rate is independent of the Knudsen number regardless of the value of the accommodation coefficient α:

$$Q_c(\delta_o PH,\alpha) = 1 \qquad \forall\alpha, \quad \forall\delta.$$

Therefore, in this case, Eq. (17) reduces to the generalized Reynolds equation introduced by Fukui and Kaneko [1], [2], while in the continuum regime, Eq. (17) reduces to the classical Reynolds equation (12).

Writing the non-dimensional film thickness H in terms of the longitudinal coordinate X,

$$H = \frac{h_2}{h_o} - \frac{l}{L}\left(\frac{h_2}{h_o} - 1\right)X \tag{18}$$

such that

$$\frac{dP}{dX} = -\frac{l}{L}\left(\frac{h_2}{h_o} - 1\right)\frac{dP}{dH}$$

Eq. (17) can be analytically integrated to give:

$$\frac{l}{L}\left(\frac{h_2}{h_o} - 1\right)\tilde{Q}_p(\delta_o PH,\alpha_1,\alpha_2)PH^3\frac{dP}{dH} + Q_c(\delta_o PH,\alpha_1,\alpha_2)\Lambda PH = K_1 \tag{19}$$

where K_1 is a constant of integration. The substitution of

$$PH = \zeta \tag{20}$$

in Eq. (19) gives:

$$\frac{d\zeta}{dH} = \frac{\zeta}{H} - \frac{[Q_c(\delta_o\zeta,\alpha_1,\alpha_2)\Lambda\zeta - K_1]}{l/L(h_2/h_o - 1)\tilde{Q}_p(\delta_o\zeta,\alpha_1,\alpha_2)H\zeta} \tag{21}$$

Eq. (21) has been solved numerically using the relaxation methods. To apply this numerical scheme, the differential equations have to be replaced by finite-difference equations on a point mesh. The solution of the resulting set of equations is determined by starting with a guess and improving it iteratively using Newton's method. The Poiseuille flow rate coefficient $Q_p(\delta,\alpha_1,\alpha_2)$ has been evaluated by means of the numerical method described in [3] and a variational technique which applies to the integrodifferential form of the Boltzmann equation based on the BGK model [4]. In order to compute the Couette flow rate $Q_c(\delta,\alpha_1,\alpha_2)$ we have solved numerically Eq. (9), extending a finite difference technique first introduced by Cercignani and Daneri [7].
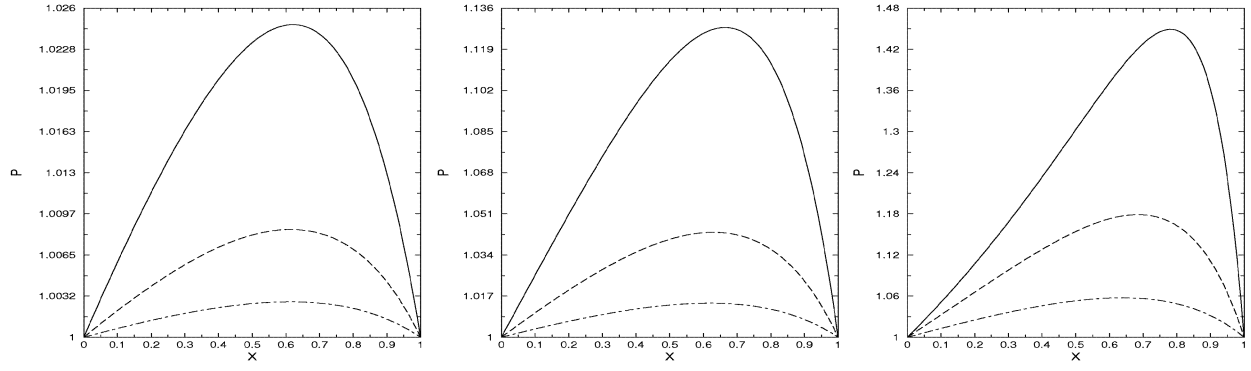
FIGURE 2. Pressure profile for $\delta_o = 0.5$. The line styles indicate $\alpha = 0.8$ (solid), $\alpha = 0.3$ (dashed), and $\alpha = 0.1$ (dot dashed). From left to right, the bearing number Λ is 10, 50, 200.
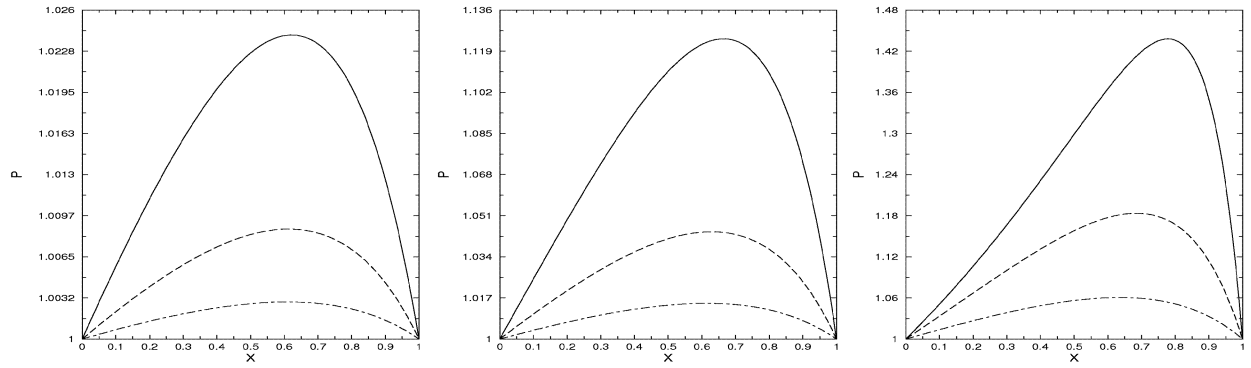

FIGURE 3. Pressure profile for $\delta_o = 0.5$. The line styles indicate $\alpha_1 = 0.5$ $\alpha_2 = 0.8$ (solid), $\alpha_1 = 0.5$ $\alpha_2 = 0.3$ (dashed), and $\alpha_1 = 0.5$ $\alpha_2 = 0.1$ (dot dashed). From left to right, the bearing number Λ is 10, 50, 200.

NUMERICAL RESULTS OF LUBRICATION CHARACTERISTICS

Once $\zeta(H)$ has been numerically evaluated on a grid that spans the domain of interest, Eqs. (18) and (20) give the pressure field in the gas film as a function of X. Furthermore, a prediction of the vertical force acting on the upper surface of the slider bearing, crucial for practical design, may be obtained from the load carrying capacity W, defined as

$$W = \frac{l}{L} \int_0^{L/l} (P - 1) \, dX \tag{22}$$

In order to investigate the effects of the rarefaction parameter δ_o and the bearing number Λ on the basic lubrication characteristics (pressure distribution and load carrying capacity), the parameters describing the gas film geometric configuration were fixed at the following values: $h_2/h_o = 2$, $L/h_o = 100$. Figure 2 shows the pressure field as a function of the longitudinal coordinate X at three different bearing numbers: $\Lambda = 10, 50, 200$. To assess the influence of the boundary conditions, the profiles corresponding to different accommodation coefficients (for bounding walls supposed physically identical) are drawn in each panel of Fig. 2. Looking at the pictures, one sees that the pressure distribution in the gas film increases with increasing Λ. Furthermore, at fixed bearing number, the pressure field reduces by increasing the fraction of gas molecules specularly reflected by the walls. Fig. 3 reports the pressure profiles for the same parameters as in Fig. 2, except that now the two bounding plates are allowed to re-emit the impinging gas molecules differently, so that two accommodation coefficients can be defined. We keep the accommodation coefficient of the upper wall (α_1) fixed and vary the other one (α_2). A comparison with Fig.2 shows that, for every Λ, the pressure distribution significantly depends on α_2 and only weakly on α_1. This picture remains true at different Knudsen numbers
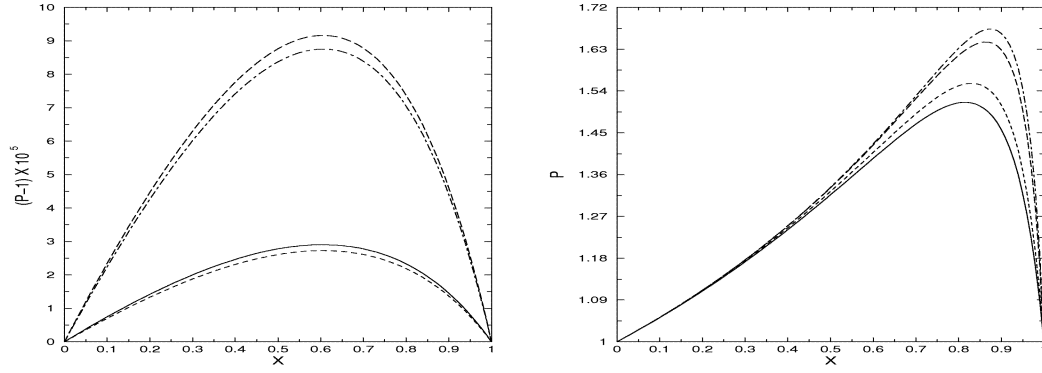
FIGURE 4. Pressure profile for $\Lambda = 50$. The line styles indicate $\alpha = 0.8$ (dot dashed), $\alpha_1 = 0.5 \; \alpha_2 = 0.8$ (long dashed), $\alpha = 0.3$ (solid), and $\alpha_1 = 0.5 \; \alpha_2 = 0.3$ (dashed). The inverse Knudsen number δ_o is 10^{-3} (left) and 10 (right).
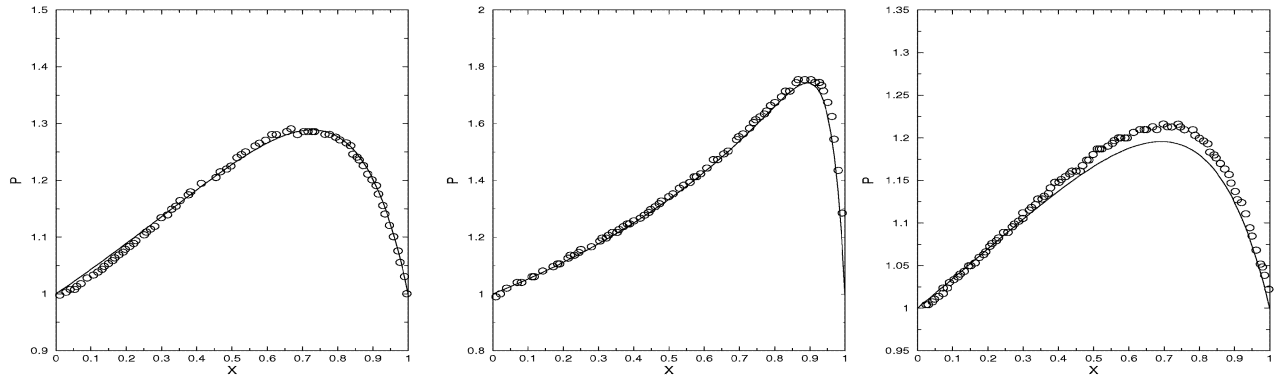

FIGURE 5. Pressure profile versus X. Comparison between our results (solid line) and DSMC data [10] (open circles). From left to right, the parameters are: (a) $\delta_o = 0.7$, $\Lambda = 61.6$, $\alpha = 1$; (b) $\delta_o = 0.2$, $\Lambda = 1264$, $\alpha = 1$; (c) $\delta_o = 0.7$, $\Lambda = 61.6$, $\alpha = 0.7$.

progressing from free molecular, through transitional, to continuum regions (see Fig.4). It is worth noting that, when Λ increases, the Couette contribution to the lubrication flow rate becomes dominant compared with the Poiseuille flow. Therefore, if the two walls are identical, the influence of the Knudsen number on the load carrying capacity decreases as Λ increases, since Q_c is independent of δ and α. On the contrary, if the two walls have a different physical structure the load carrying capacity shows a dependence on both the Knudsen number and the accommodation coefficients α_1, α_2. For the validation of the code, our outputs have been compared with the results from DSMC (Direct Simulation Monte Carlo) simulations published by Alexander et al. [10] in the case of Maxwell's boundary conditions on two physically identical walls (see Fig.5).

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